Lesson 1

Light-matter interaction and Dispersion

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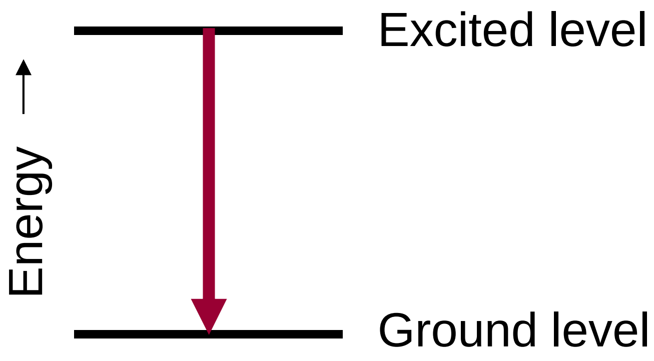
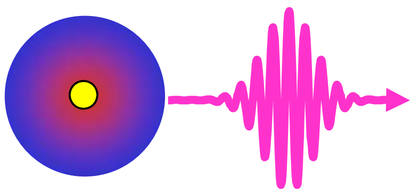
Various slides under courtesy of Prof. R. Trebino at GIT

- Light-matter interaction
  - Absorption of light
    - Forced oscillator model
  - Reemission of light
  - Frequency dependent phase

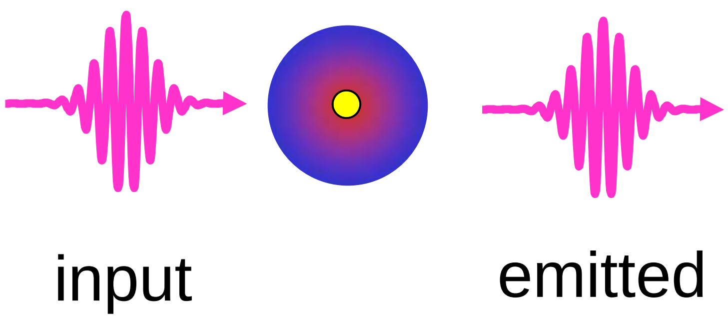
The interaction of light and matter

Absorption, resonance: gives everything we see.



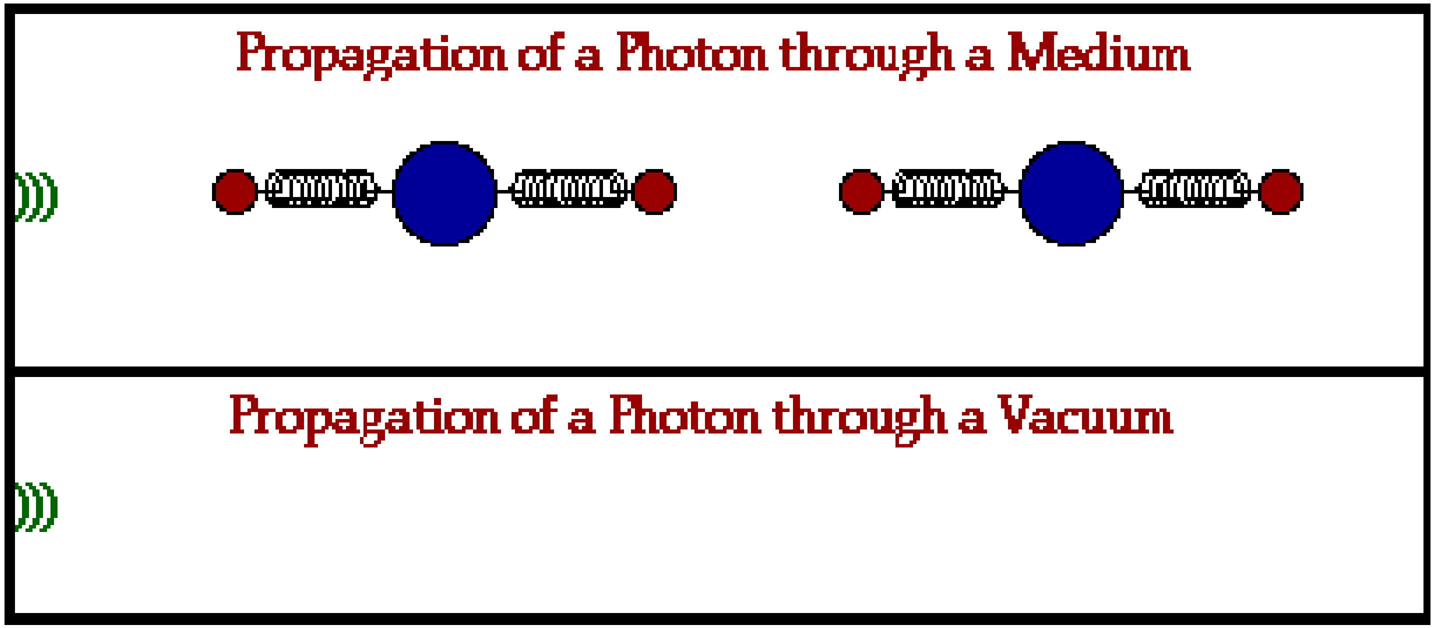


Light vibrates matter➔ Matter emits light➔ interferes with the original light.



Light propagation in medium

- Absorption and Re-emission of the wave energy by the atoms:
  - Initially, the energy of the wave is absorbed by the atom.
  - This energy causes the electrons within the atoms to undergo vibrations.
  - After a short vibrational motion, the electrons create a new electromagnetic wave with the same frequency but with different phase
  - Waves propagate at slower speed in medium than in vacuum.



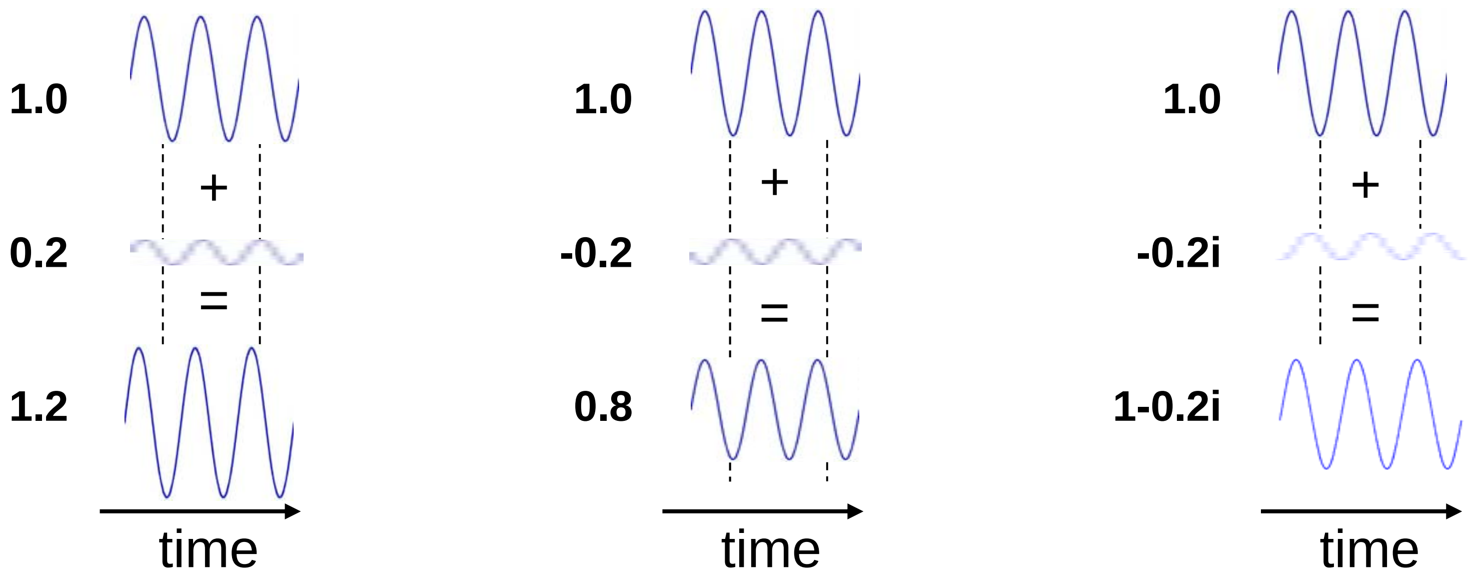
http://www.physicsclassroom.com/mmedia/waves/em.cfm

Phasor addition

That's why *coherence* matters!!

When two waves add together with the same exponentials, we add the complex amplitudes, $E_0 + E_0'$.

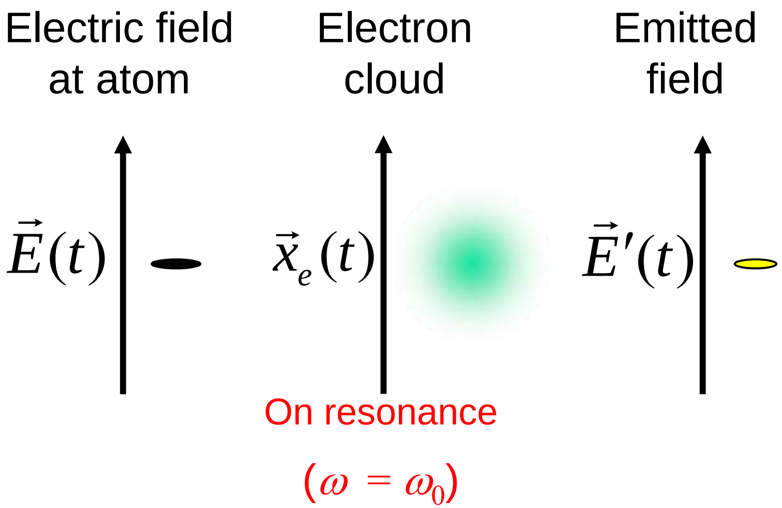
Constructive interference: | Destructive interference: | "Quadrature phase" $\pm 90^\circ$ interference:

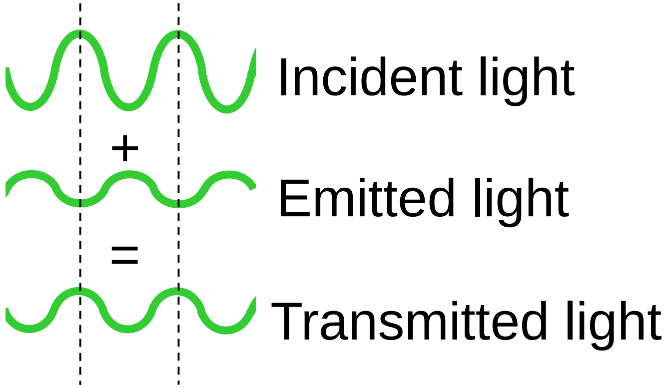


Laser | Absorption | Slower phase velocity

Light ➔ Atom ➔ Light

When light of frequency ω excites an atom with resonant frequency ω_0:





An excited atom vibrates at the frequency of the light that excites it and re-emits the energy as light of that frequency.

The crucial issue is the **relative phase** of the incident light and this re-emitted light. For example, if these two waves are ~180° out of phase, the beam will be attenuated. We call this absorption.

The forced oscillator

When we apply a periodic force to a natural oscillator (such as a pendulum, spring, swing, or atom), the result is a **forced oscillator**.

Tacoma Narrows Bridge (1940) collapsing because oscillatory winds blew at its resonance frequency.

Examples:

- Child on a swing being pushed
- Pushed pendulum
- Bridge in wind or an earthquake
- Electron in a light wave
- Nucleus in a light wave

The forced oscillator is one of the most important problems in physics. It is the concept of **resonance**.

The forced oscillator: math

Consider an electron on a spring with position $x_e(t)$, and driven by a light wave, $E_0 \exp(j\omega t)$:

$$\boxed{m_e d^2 x_e / dt^2 \; + \; m_e \omega_0^2 x_e \; = \; qE_0 \exp(j\omega t)}$$

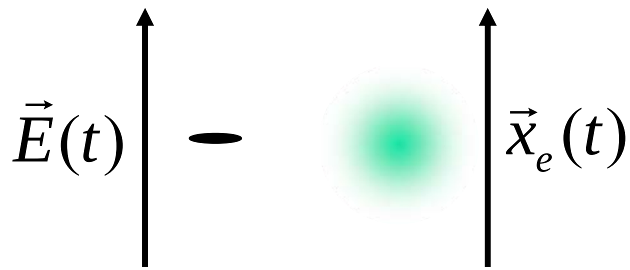
restoring force

depends on displacement

The solution is:

$$\boxed{x_e(t) = \left[\frac{(q/m_e)}{(\omega_0^2 - \omega^2)} \right] E_0 \exp(j\omega t)}$$

Phase depends on sign of charge



So the electron oscillates at the incident light wave frequency (ω), but with a frequency-dependent amplitude.

Checking our solution

$$\boxed{m_e d^2 x_e / dt^2 \; + \; m_e \omega_0^2 x_e \; = \; qE_0 \exp(j\omega t)}$$

Substitute the solution for $x_e(t)$ into the forced oscillator equation to see if it works.

$$\boxed{x_e(t) = \left[\frac{(q/m_e)}{\left(\omega_0^2 - \omega^2\right)} \right] E_0 \exp(j\omega t)}$$

$$-m_e\omega^2 \left[\frac{(q/m_e)}{\left(\omega_0^2 - \omega^2\right)} \right] \cancel{E_0 \exp(j\omega t)} + m_e\omega_0^2 \left[\frac{(q/m_e)}{\left(\omega_0^2 - \omega^2\right)} \right] \cancel{E_0 \exp(j\omega t)} = q\cancel{E_0 \exp(j\omega t)}$$

$$-\cancel{m_e}\omega^2 \left[\frac{(\cancel{q/m_e})}{\left(\omega_0^2 - \omega^2\right)} \right] + \cancel{m_e}\omega_0^2 \left[\frac{(\cancel{q/m_e})}{\left(\omega_0^2 - \omega^2\right)} \right] = \cancel{q}$$

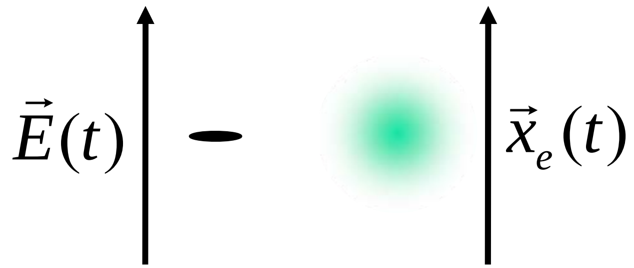
$$-\omega^2 \left[\frac{1}{\left(\omega_0^2 - \omega^2\right)} \right] + \omega_0^2 \left[\frac{1}{\left(\omega_0^2 - \omega^2\right)} \right] = 1$$

$$\left[\frac{\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)} \right] = 1$$

The problem with this model

$$x_e(t) = \left[\frac{(q/m_e)}{(\omega_0^2 - \omega^2)} \right] E_0 \exp(j\omega t)$$

Exactly on resonance, when $\omega = \omega_0$, x_e goes to **infinity**.



This is unrealistic (**op-amp**)

We'll need to fix this.

The damped forced oscillator

Our solution has infinite amplitude on resonance, which is unrealistic. We fix this by using a damped forced oscillator: a harmonic oscillator experiencing a sinusoidal force and viscous drag.

We must add a viscous drag term: $m_e \gamma \frac{dx_e}{dt}$

depends on velocity

$$\boxed{m_e \frac{d^2 x_e}{dt^2} + m_e \gamma \frac{dx_e}{dt} + m_e \omega_0^2 x_e = qE_0 \exp(j\omega t)}$$

The solution is now:

$$\boxed{x_e(t) = \left[\frac{(q/m_e)}{(\omega_0^2 - \omega^2 + j\omega\gamma)} \right] E(t)}$$

The electron still oscillates at the light frequency and with a potential **phase shift**, but now with a finite amplitude for all ω.

Why we include the damping factor, γ ?

Atoms spontaneously decay to the ground state after a time.

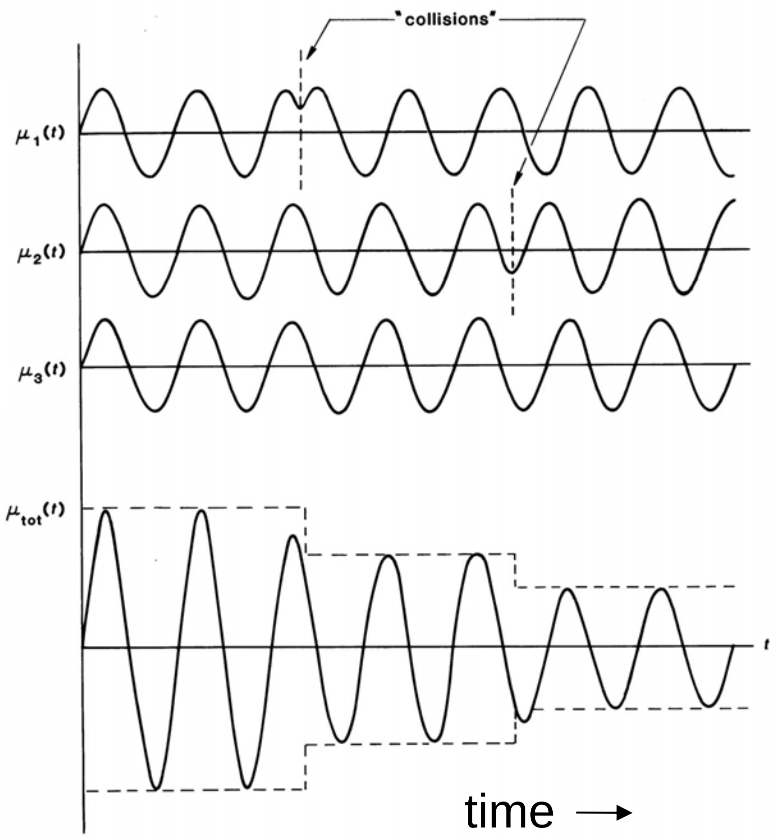
Also, the vibration of a medium is the sum of the vibrations of all the atoms in the medium, and collisions cause the sum to cancel.

Atom #1

Atom #2

Atom #3

Sum:



Collisions **dephase** the vibrations, causing cancellation of the total medium vibration, typically exponentially.

(We can use the same argument for the emitted light, too.)

Recall coherence and linewidth!

Complex Lorentzian approximation

Consider: $$x_e(t) \propto \frac{q/m_e}{\omega_0^2 - \omega^2 + j\omega\gamma} = \frac{q/m_e}{(\omega_0 + \omega)(\omega_0 - \omega) + j\omega\gamma}$$

Assuming $\omega \approx \omega_0$, this becomes: $$= \frac{q/m_e}{2\omega(\omega_0 - \omega) + j\omega\gamma}$$

$$= \frac{q/m_e}{2\omega} \frac{1}{(\omega_0 - \omega) + j\gamma/2}$$

In terms of the variables $\boldsymbol{\delta = \omega_0 - \omega}$ and $\Gamma = \gamma/2$, the function $1/(\delta + j\Gamma)$, is called a **Complex Lorentzian**. Its real and imaginary parts are:

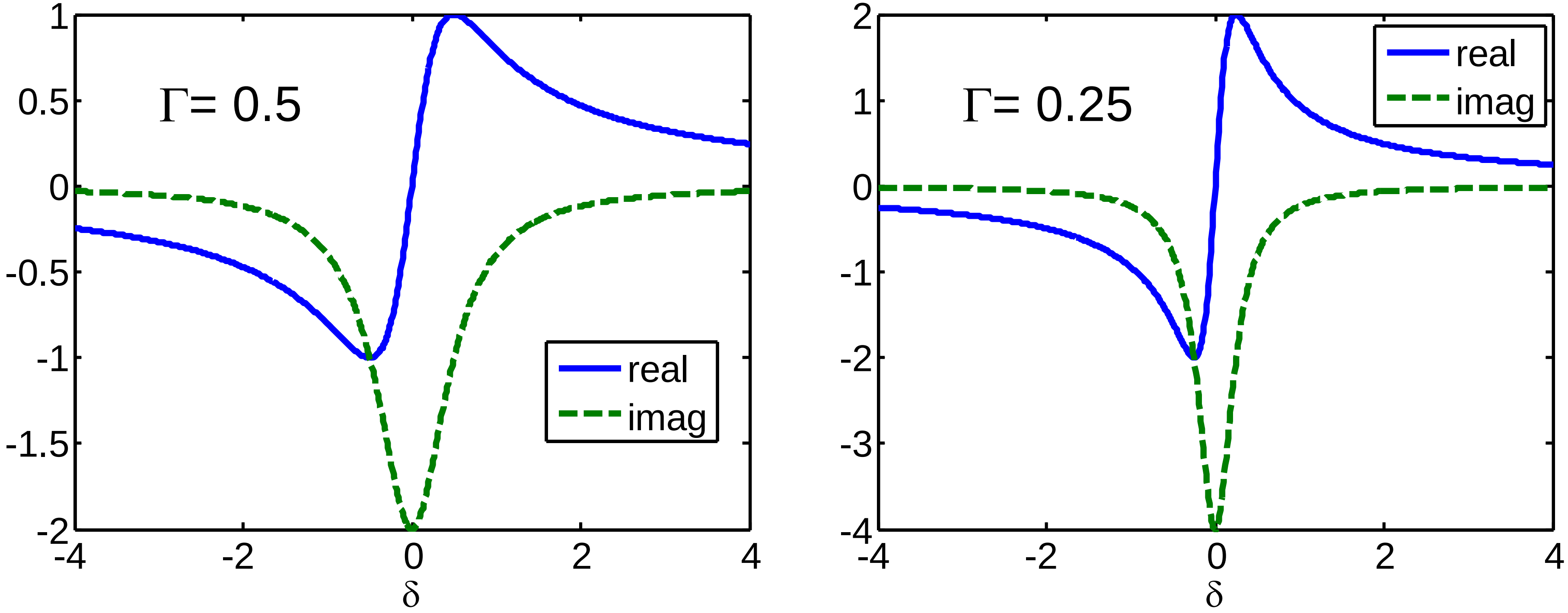
$$\frac{1}{\delta + j\Gamma} = \frac{1}{\delta + j\Gamma}\frac{\delta - j\Gamma}{\delta - j\Gamma} = \frac{\delta}{\delta^2 + \Gamma^2} - j\frac{\Gamma}{\delta^2 + \Gamma^2}$$

Complex Lorentzian

Real Imaginary

$$\frac{1}{\delta + j\Gamma} = \frac{\delta}{\delta^2 + \Gamma^2} - j\frac{\Gamma}{\delta^2 + \Gamma^2}$$

smaller Γ, narrower linewidth



Damped forced oscillator for light-driven atoms

The forced-oscillator response is sinusoidal, with a frequency-dependent strength that's approximately a complex Lorentzian:

Here, q < 0.

$$\boxed{x_e(t) \approx \left(\frac{q}{2\omega m_e}\right)\left[\frac{1}{(\omega_0 - \omega + j\gamma/2)}\right]E(t) \propto -\left[\frac{1}{(\omega_0 - \omega + j\gamma/2)}\right]E(t)}$$

When $\omega << \omega_0$, the electron vibrates 180° out of phase with the light wave:

$$x_e(t) \propto -\left[\frac{1}{(\omega_0)}\right]E(t) \propto -E(t)$$

When $\omega = \omega_0$, the electron vibrates 90° out of phase with the light wave:

$$x_e(t) \propto -\left[\frac{1}{(j\gamma/2)}\right]E(t) \propto jE(t)$$

When $\omega >> \omega_0$, the electron vibrates in phase with the light wave:

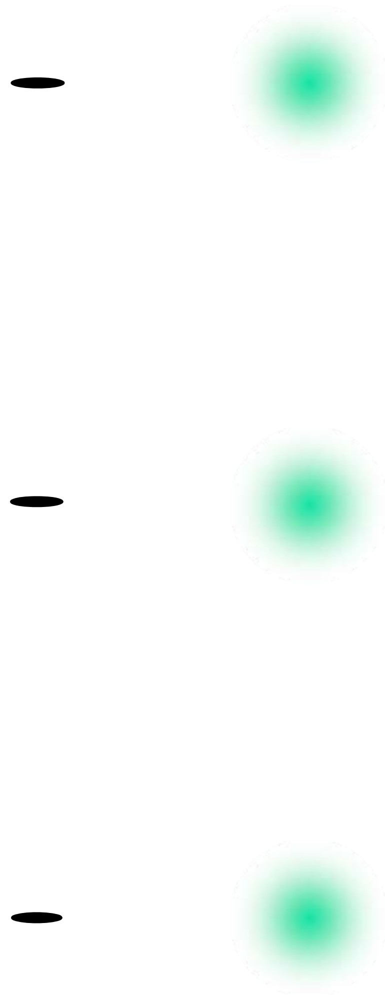
$$x_e(t) \propto -\left[\frac{1}{(-\omega)}\right]E(t) \propto E(t)$$

The electron cloud

The relative phase of an electron cloud's motion with respect to input light depends on the frequency.

Recall that the atom's resonant frequency is ω_0, and the light frequency is ω.

	Electric field at atom	Electron cloud	
Below resonance $\omega << \omega_0$			**Weak vibration 180° out of phase**
On resonance $\omega = \omega_0$			**Strong vibration 90° out of phase** **(270° phase lag)**
Above resonance $\omega >> \omega_0$			**Weak vibration in phase**



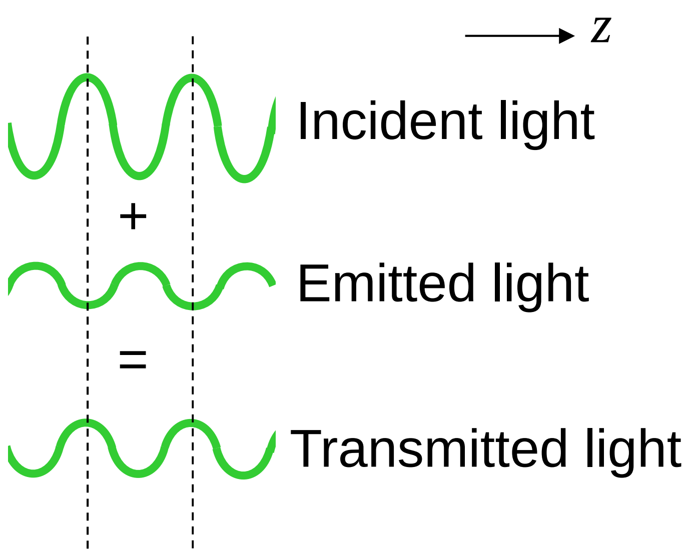
Okay, so now we know what the lightwave does to the atom.

But, what does the atom do to the lightwave?

Re-emitted light from an excited atom

The re-emitted light may interfere constructively, destructively, or, more generally, somewhat out of phase with the original light wave.

We model this process by considering the total electric field,



$$\boxed{E(z,t) = E_{original}(z,t) + E_{re\text{-}emitted}(z,t)}$$

Maxwell's Equations will allow us to solve for the **total** field, $E(z,t)$. The input field will be the initial condition.

How to take medium into account?

- In vacuum $$\vec{D} = \varepsilon_0 \vec{E}$$

- Now that we have polarization $$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \qquad \boxed{\vec{P}(t) = Nq\vec{x}_e(t)}$$

- If polarization is taken as scalar $$P^{(1)} = \varepsilon_0 \chi^{(1)} E$$

$$\boxed{D = \varepsilon_0 (1 + \chi^{(1)}) E}$$

So, if you want to see the effect of the medium
you need to use the electric field density D.

Maxwell's Equations for a Medium

- The induced polarization, $\vec{P}$, contains the effect of the medium and is included in Maxwell's Equations:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}+\mu_0\frac{\partial\vec{P}}{\partial t}$$

This extra term turn it into the **Inhomogeneous Wave Equation**:

$$\frac{\partial^2\vec{E}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=\mu_0\frac{\partial^2\vec{P}}{\partial t^2}$$

The polarization is the **driving/source term** and tells us what light will be emitted.

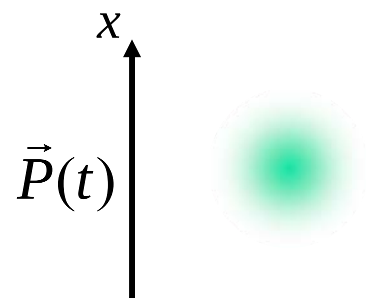
The inhomogeneous wave equation

The induced polarization, $\vec{P}$, is due to the medium:

$$\boxed{\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}}$$

where:

$$\vec{P}(t) = Nq\vec{x}_e(t)$$



$$x_e(t) \approx \left(\frac{q}{2\omega m_e}\right)\left[\frac{1}{(\omega_0 - \omega + j\gamma/2)}\right]E(t)$$

For our vibrating electrons:

$$P(z,t) \approx \underbrace{Nq\left(\frac{q}{2\omega m_e}\right)\left[\frac{1}{(\omega_0 - \omega + j\gamma/2)}\right]}_{\equiv P_0}\underbrace{E_0 \exp[j(\omega t - kz)]}_{E(z,t)}$$

The electric-field amplitude depends on z

The effect of the medium is to change the field complex amplitude with distance. And because the polarization depends on E, its amplitude, P_0, will also.

Constant in time

$$E(z,t) = E_0(z)\exp\left[j\left(\omega t - kz\right)\right] \text{ and } P(z,t) = P_0(z)\exp\left[j\left(\omega t - kz\right)\right]$$

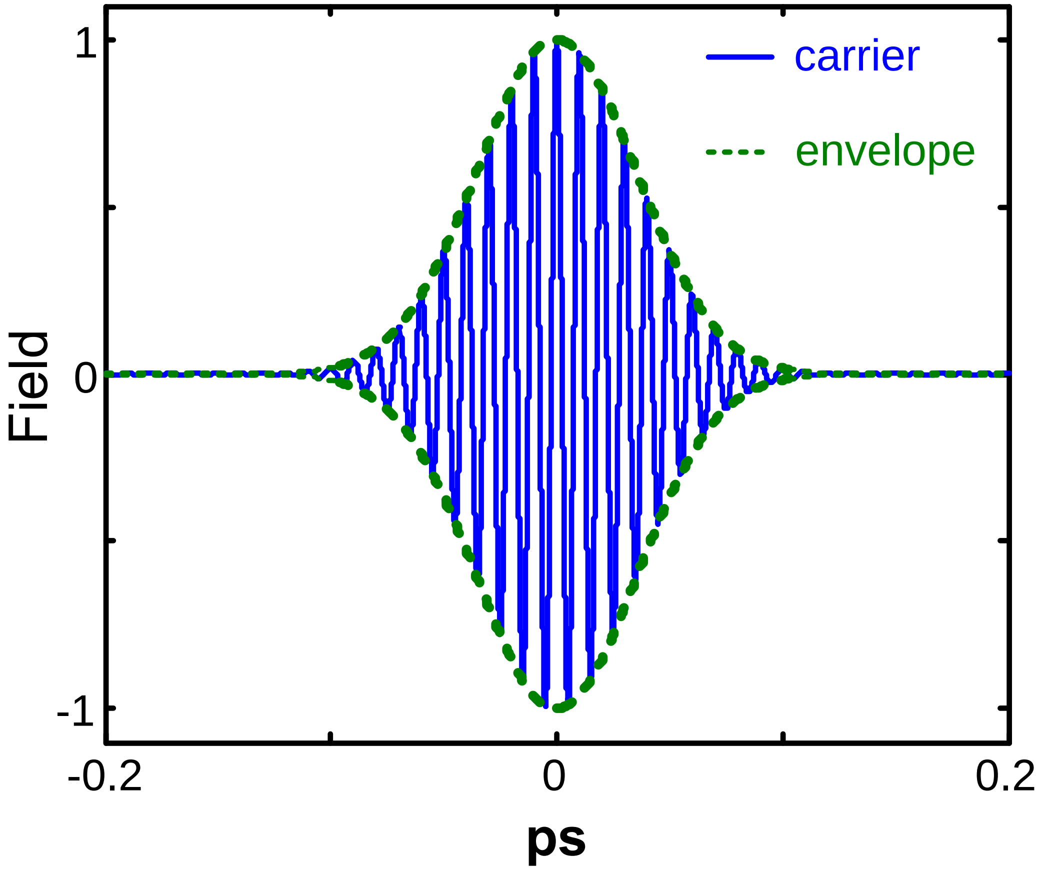
Specifically, the envelopes, $E_0(z)$ and $P_0(z)$, are assumed to vary **slowly**; the fast variations will all be in the complex exponential.

The time derivatives are easy (as before, they just multiply by a factor of $-\omega^2$) because the envelopes are independent of t:

$$-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\omega^2}{c^2}E \qquad\qquad \mu_0\frac{\partial^2 P}{\partial t^2} = -\mu_0\omega^2 P$$

Pulse envelope and carrier

$$E(t) = \mathrm{Re}\left\{\underbrace{A(t)}_{\text{envelope}}\underbrace{\exp^{j\omega t}}_{\text{carrier}}\right\}$$



The Slowly Varying Envelope Approximation

But the $\partial^2 E / \partial z^2$ derivative is trickier.

$$E(z,t) = E_0(z)\exp\left[j\left(\omega t - kz\right)\right]$$

The z-derivatives:

$$\frac{\partial E(z,t)}{\partial z} = \left[\frac{\partial E_0}{\partial z} - jkE_0(z) \right] \exp\left[j\left(\omega t - kz\right)\right]$$

$$\frac{\partial^2 E(z,t)}{\partial z^2} = \left(\frac{\partial^2 E_0}{\partial z^2} - jk\frac{\partial E_0}{\partial z} \right) \exp\left[j\left(\omega t - kz\right)\right] - jk\left(\frac{\partial E_0}{\partial z} - jkE_0 \right) \exp\left[j\left(\omega t - kz\right)\right]$$

$$\Rightarrow \quad \frac{\partial^2 E(z,t)}{\partial z^2} = \left[\frac{\partial^2 E_0}{\partial z^2} - 2jk\frac{\partial E_0}{\partial z} - k^2 E_0 \right] \exp\left[j\left(\omega t - kz\right)\right]$$

Because variations of the envelope, $E_0(z)$, in space will be slow, we'll neglect the 2nd z-derivative.

SVEA continued

Substituting the derivatives into the inhomogeneous wave equation:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$\left[-2jk\frac{\partial E_0}{\partial z} - k^2 E_0 + \frac{\omega^2}{c^2}E_0\right]\exp\left[j\left(\omega t - kz\right)\right] = -\mu_0\omega^2 P_0 \exp\left[j\left(\omega t - kz\right)\right]$$

Benefit using carrier-envelope

Now, use $k = \omega/c$. And canceling the complex exponentials leaves:

$$-2jk\frac{\partial E_0}{\partial z} = -\mu_0\omega^2 P_0$$

$$\Rightarrow \quad \frac{\partial E_0}{\partial z} = \frac{\mu_0\omega^2}{2jk}P_0 = \frac{\mu_0(k^2/\mu_0\varepsilon_0)}{2jk}P_0 = -j\frac{k}{2\varepsilon_0}P_0$$

Re-emitted light is 90° out of phase with P

$$\boxed{\frac{\partial E_0}{\partial z} = -j\frac{k}{2\varepsilon_0}P_0}$$

Usually, $P_0 = P_0\,(E_0)$, and hence $P_0(z)$, too.
But consider for the moment $P_0 \sim$ constant.

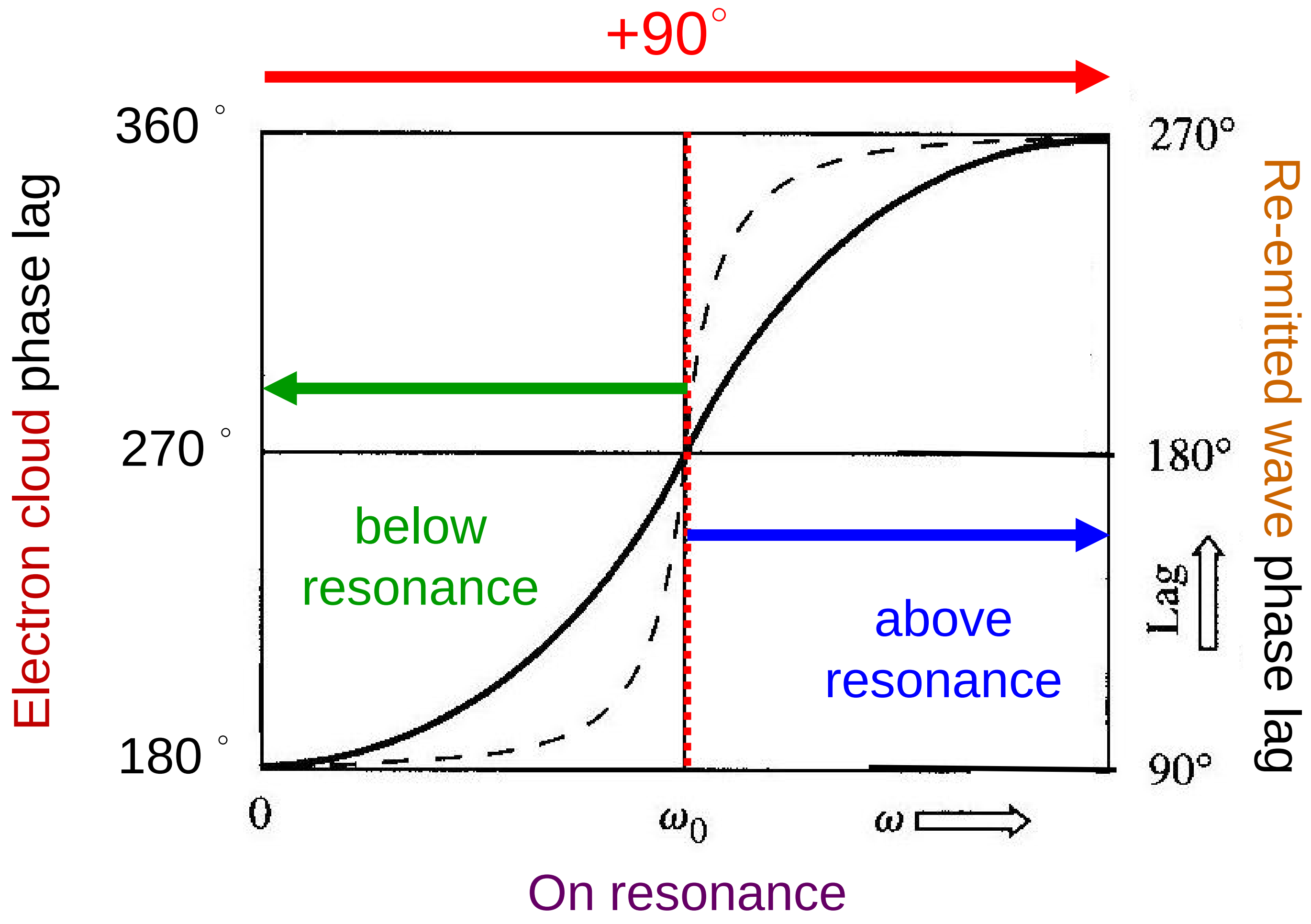
Converting to finite differences, the re-emitted field is just ΔE_0, and taking the negative charge of electron into consideration, it will be:

$$\Delta E_0 \approx j\frac{k}{2\varepsilon_0}\Delta z\left|P_0\right|$$

Note the j, which means that the re-emitted field has a **90° phase lead** with respect to the electron cloud motion.

Phase (frequency dependent) relation

- Input vs. oscillator
- Input vs. re-emitted light

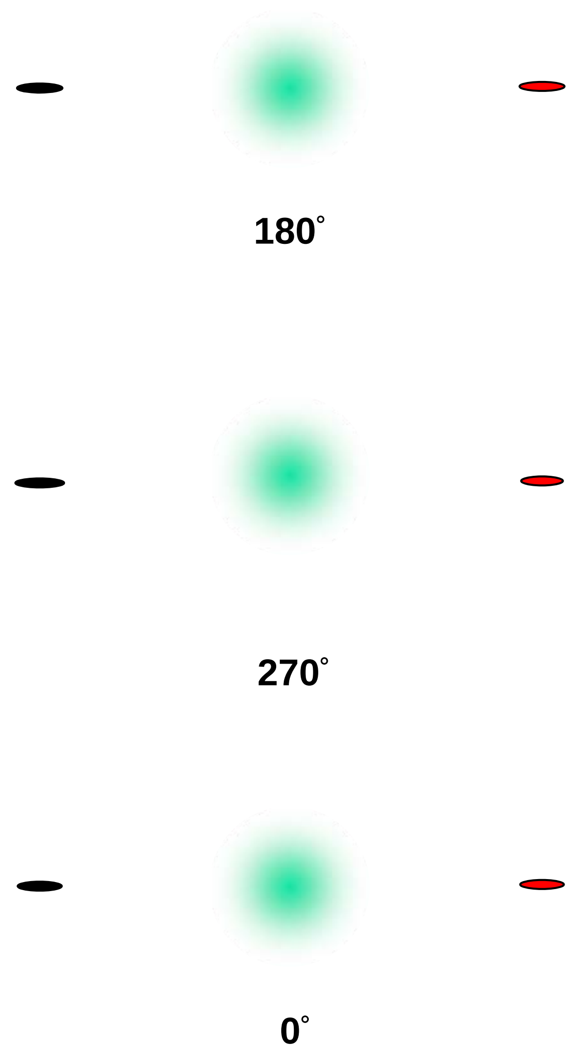


The entire process

The re-emitted wave leads the electron cloud motion by 90°

This phase shift adds to the potential phase shift of the electron cloud motion with respect to the input light.

	Electric field at atom	Electron cloud	Emitted field	
Below resonance $\omega << \omega_0$				**Weak emission 90° out of phase**
On resonance $\omega = \omega_0$				**Strong emission 180° out of phase**
Above resonance $\omega >> \omega_0$				**Weak emission -90° out of phase**



- Dispersion
  - Complex Lorentzian
  - Real part: $n(\omega)$
  - Imaginary part: absorption

Solving for the slowly varying envelope

Define χ, the **susceptibility**:

$$P_0 = Nq\left(\frac{q}{2\omega m_e}\right)\left[\frac{1}{(\omega_0 - \omega + j\gamma/2)}\right]E_0 \equiv \varepsilon_0 \chi E_0$$

$$\chi = \frac{Nq}{\varepsilon_0}\left(\frac{q}{2\omega m_e}\right)\left[\frac{1}{(\omega_0 - \omega + j\gamma/2)}\right]$$

$$= \frac{Nq^2}{2\varepsilon_0 \omega m_e}\left[\frac{\delta}{\delta^2 + \Gamma^2} + j\left(\frac{-\Gamma}{\delta^2 + \Gamma^2}\right)\right]$$

$\delta = \omega_0 - \omega$
and $\Gamma = \gamma/2$

$$= \mathrm{Re}\{\chi\} + j\,\mathrm{Im}\{\chi\}$$

The wave undergoes attenuation

- Attenuation coefficient α
- Refractive index (*n-1*)

$$\chi = \frac{Nq^2}{2\varepsilon_0 \omega m_e}\left[\frac{\delta}{\delta^2+\Gamma^2} + j\left(\frac{-\Gamma}{\delta^2+\Gamma^2}\right)\right]$$

$$= \mathrm{Re}\{\chi\} + j\,\mathrm{Im}\{\chi\}$$

$$\frac{\partial E_0}{\partial z} = -j\frac{k}{2\varepsilon_0}P_0 \quad \Rightarrow \frac{\partial E_0}{\partial z} = -j\frac{k}{2}\chi E_0$$

The solution: $E_0(z) = E_0(0)\exp\left[-j\frac{k}{2}\chi z\right] = E_0(0)\exp\left[-j\frac{k}{2}\left(j\,\mathrm{Im}\{\chi\} + \mathrm{Re}\{\chi\}\right)z\right]$

Define new quantities for the real and imaginary parts of χ:

$$\boxed{\alpha \equiv k\,|\,\mathrm{Im}\{\chi\}\,|} \qquad \boxed{(n-1) \equiv \frac{1}{2}\mathrm{Re}\{\chi\}}$$

so that: $E_0(z) = E_0(0)\exp\left\{[-\alpha/2 - j(n-1)k]z\right\}$

where α is the **absorption coefficient** and n is the **refractive index**.

The complete electric field in a medium

The electromagnetic wave in the medium becomes (combining the slowly varying envelope with the complex exponential):

$$E(z,t) = \underbrace{E_0(0)\exp\{[-\alpha/2 - j(n-1)k]z\}}_{E_0(z)}\exp[j(\omega t - kz)]$$

Simplifying:

$$\boxed{E(z,t) = E_0(0)\exp[(-\alpha/2)z]\ \exp[j(\omega t - nkz)]}$$

Absorption attenuates the field

Refractive index changes the k-vector

To summarize, in a medium:

$$\boxed{|E_0(z)| = |E_0(0)|\exp[(-\alpha/2)z] \qquad k \rightarrow nk \quad \text{and} \quad \lambda \rightarrow \lambda/n}$$

n and α

n comes from Re $\{\chi\}$:

$$(n-1)k = \frac{k}{2}\mathrm{Re}\{\chi\} = \frac{k}{2\varepsilon_0}\mathrm{Re}\left\{\frac{Nq^2/m_e}{2\omega(\omega_0-\omega+j\gamma/2)}\right\}$$

α comes from the imaginary part of χ:

$$\alpha/2 = \frac{k}{2}\left|\mathrm{Im}\{\chi\}\right| = \frac{k}{2\varepsilon_0}\mathrm{Im}\left\{\frac{Nq^2/m_e}{2\omega(\omega_0-\omega+j\gamma/2)}\right\}$$

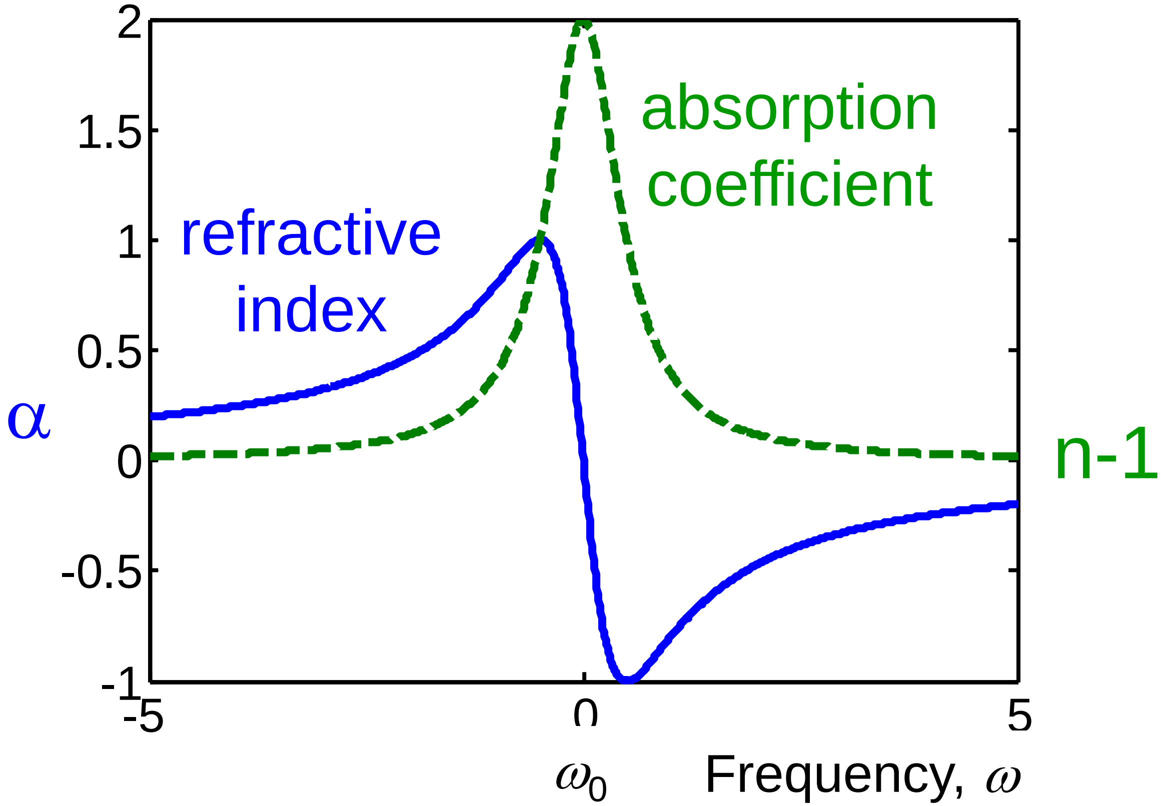
Simplifying:

$$n-1 \;=\; \frac{Nq^2}{4\varepsilon_0\omega m_e}\left[\frac{(\omega_0-\omega)}{(\omega_0-\omega)^2+(\gamma/2)^2}\right]$$

$$\alpha \;=\; \frac{Nq^2}{2\varepsilon_0 c m_e}\left[\frac{\gamma/2}{(\omega_0-\omega)^2+(\gamma/2)^2}\right]$$

These results are valid for small values of these quantities.

Refractive index and absorption coefficient

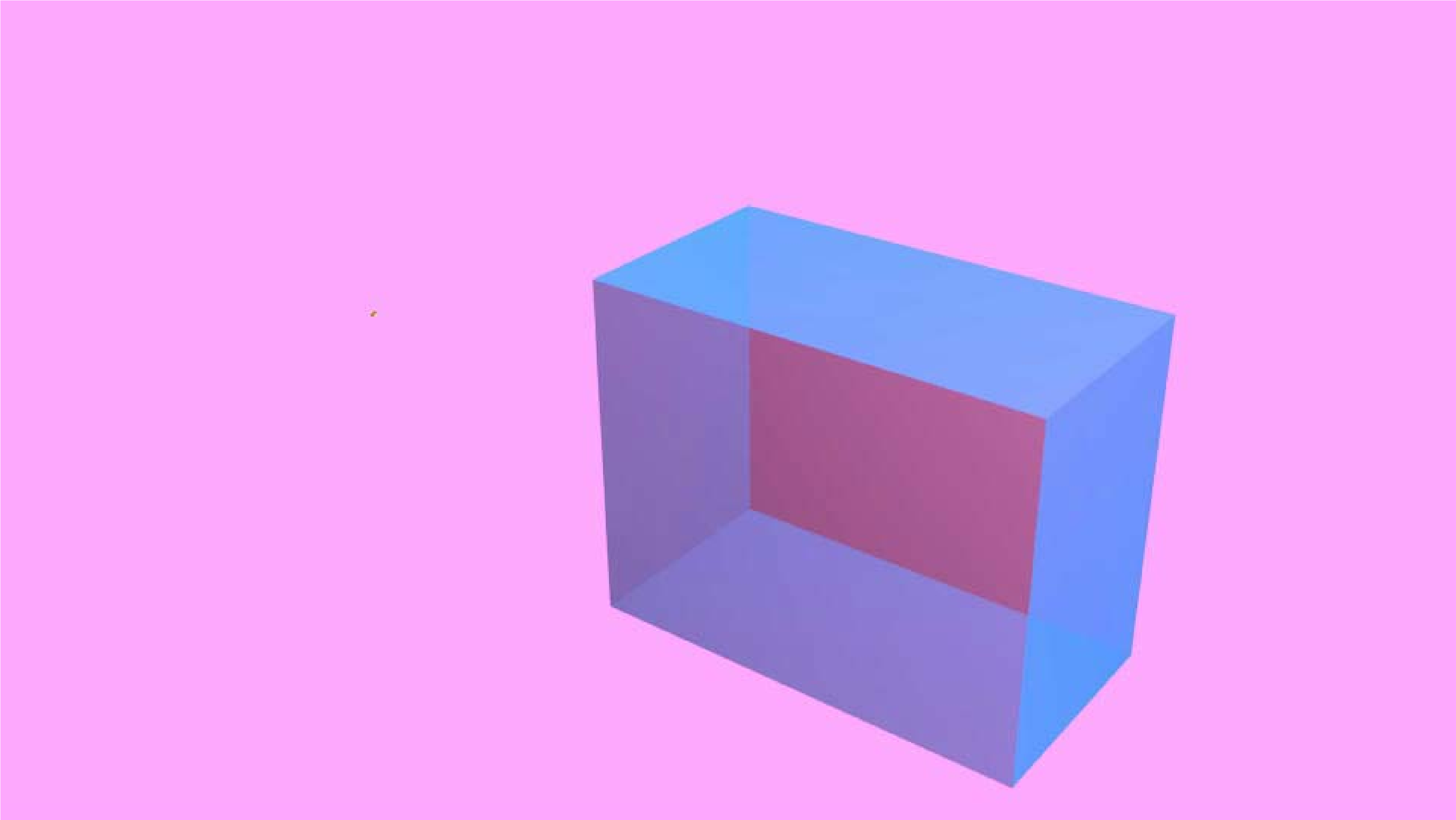


$$\alpha = \frac{Nq^2}{2\varepsilon_0 c m_e} \frac{\gamma/2}{(\omega_0-\omega)^2+(\gamma/2)^2}$$

$$n-1 = \frac{Nq^2}{4\varepsilon_0 \omega m_e} \frac{\omega_0-\omega}{(\omega_0-\omega)^2+(\gamma/2)^2}$$

Lightwave suffering attenuation

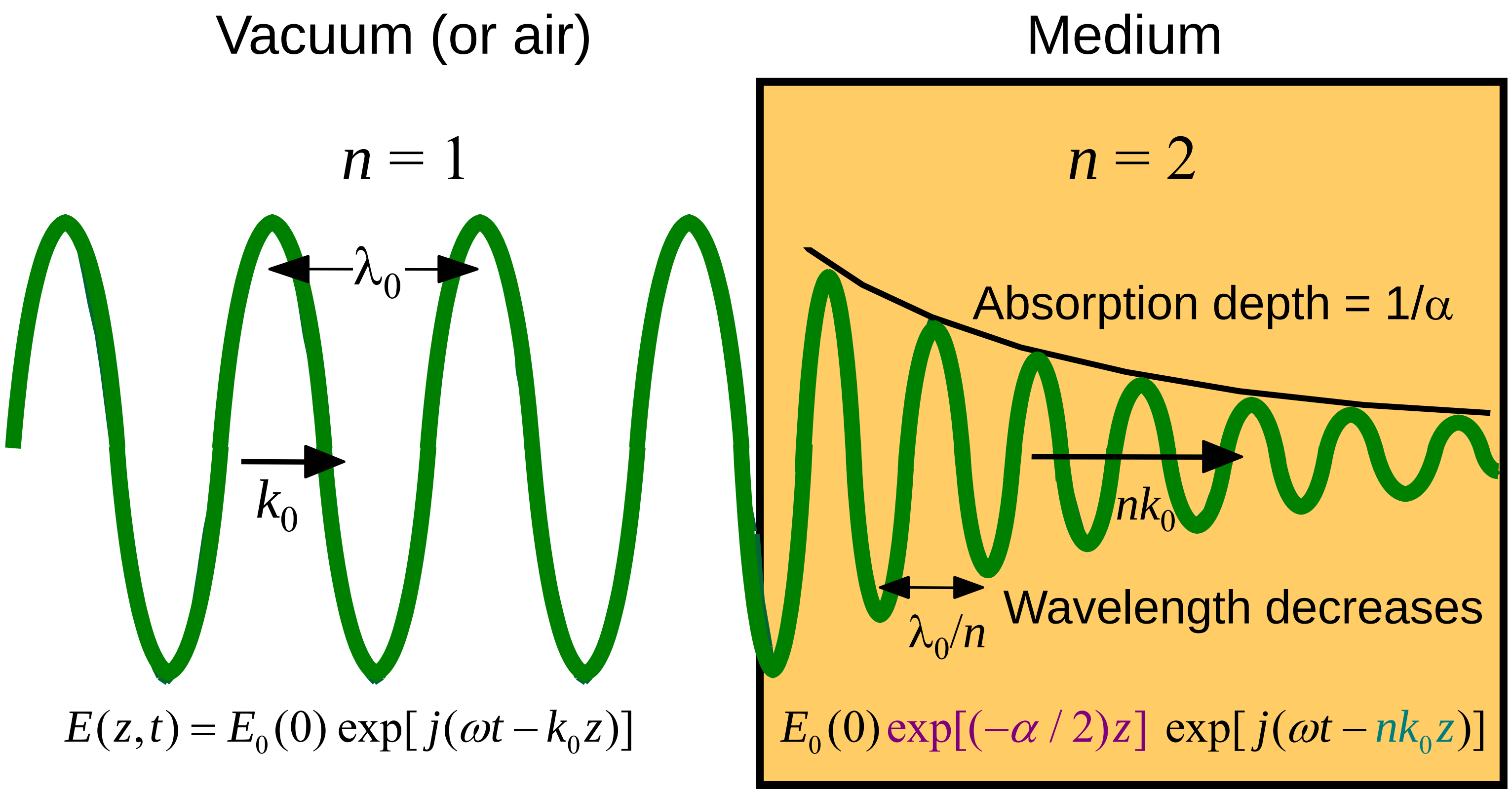
- Movie



A light wave in a medium

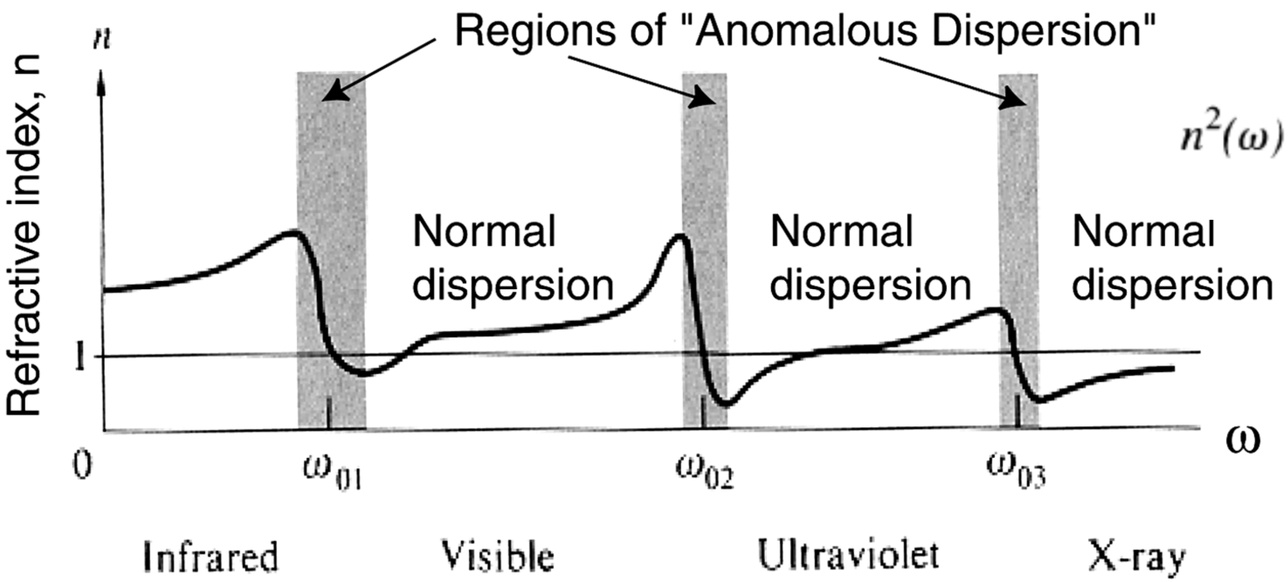
The speed of light, the wavelength (and k), and the amplitude change, but the frequency, ω, doesn't change.

$$\boxed{I(z) = I(0)\exp(-\alpha z)}$$



n(ω)

Since resonance frequencies exist in many spectral ranges, the refractive index varies in a complex manner.

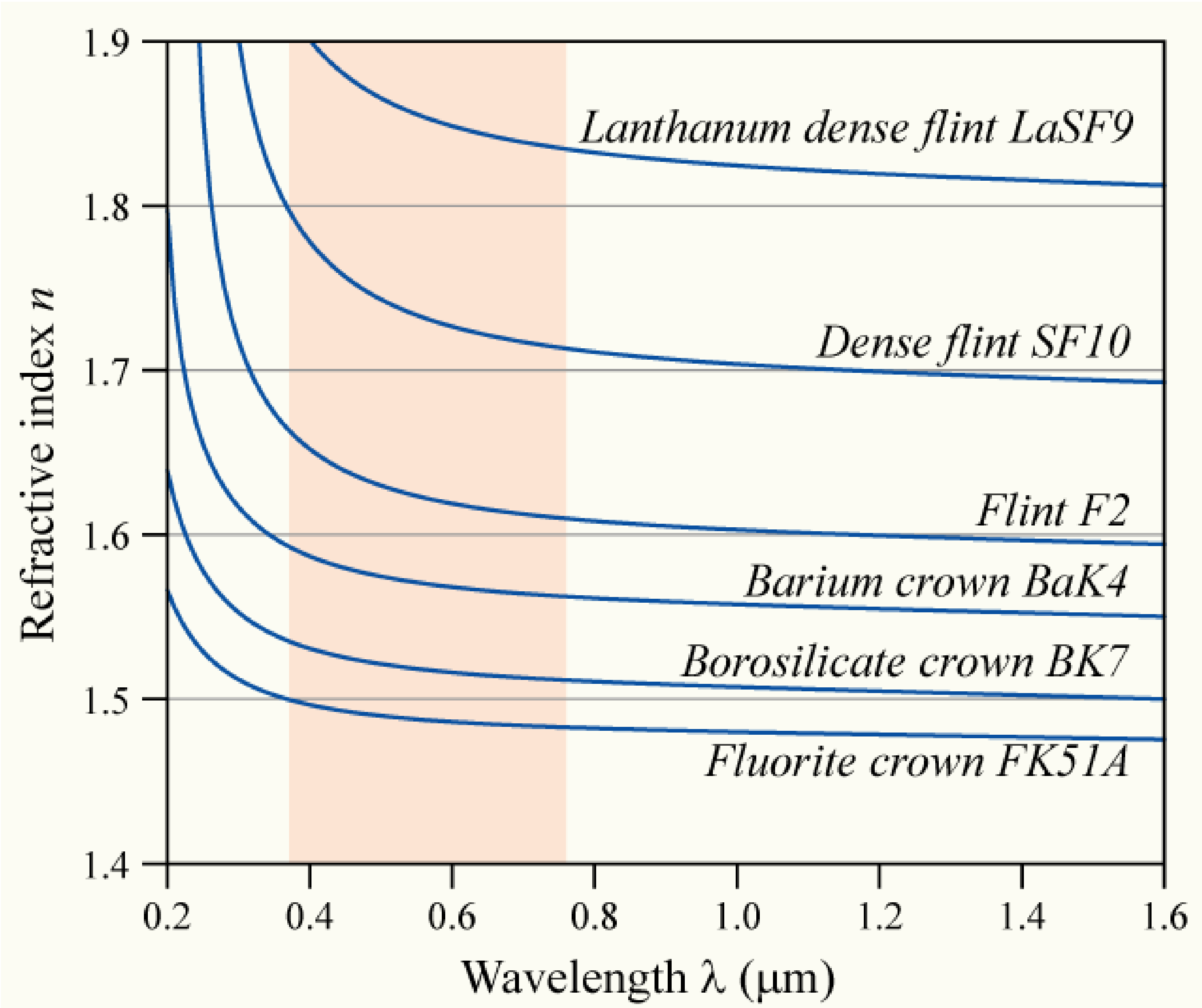


$$n^2(\omega) = 1 + \frac{N q_e^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right)$$

Electronic resonances usually occur in the UV; vibrational and rotational resonances occur in the IR; and inner-shell electronic resonances occur in the x-ray region.

n increases with frequency, except in **anomalous dispersion** regions.

Refractive indices for glasses



We'll use
$n = 1.5$
for the refractive index of the glass we usually encounter.

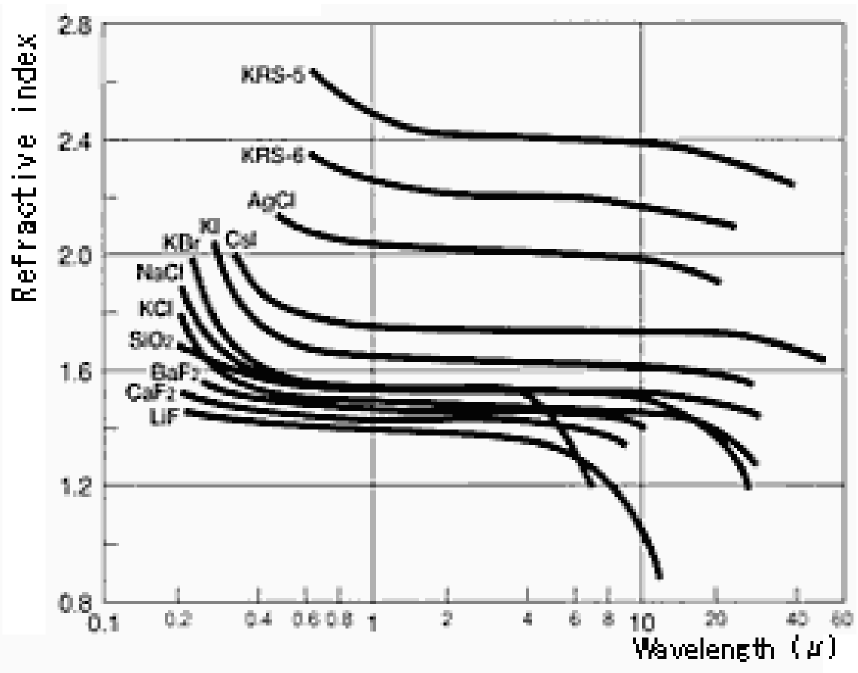
The Sellmeier Equation

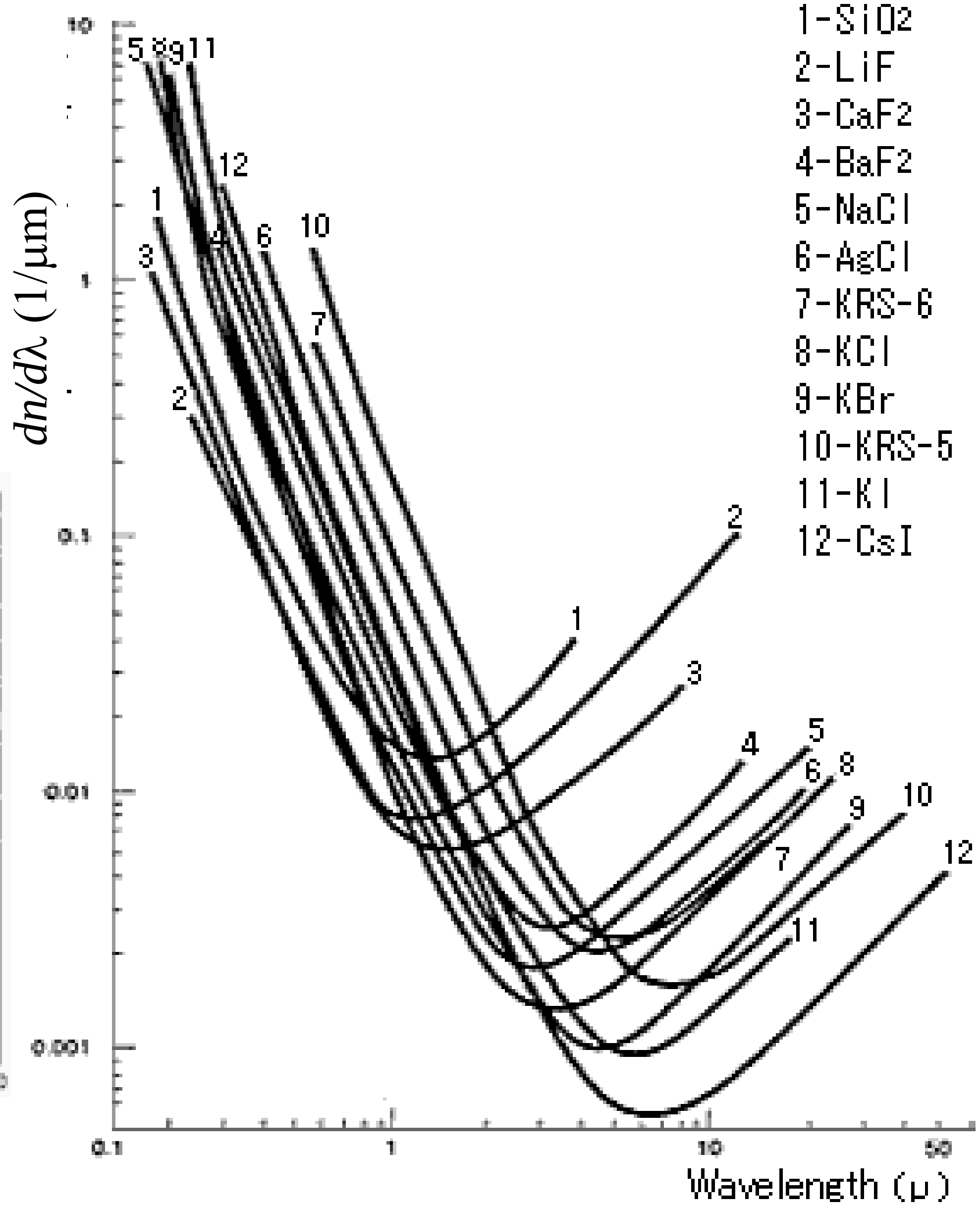
$$n^2(\lambda) = 1 + \frac{B_1\lambda^2}{\lambda^2 - C_1} + \frac{B_2\lambda^2}{\lambda^2 - C_2} + \frac{B_3\lambda^2}{\lambda^2 - C_3}$$

Coefficient	**Value**
B_1	1.03961212
B_2	2.31792344×10^{-1}
B_3	1.01046945
C_1	6.00069867×10^{-3}
C_2	2.00179144×10^{-2}
C_3	1.03560653×10^{2}

These values are obtained by measuring n for numerous wavelengths and then curve-fitting.

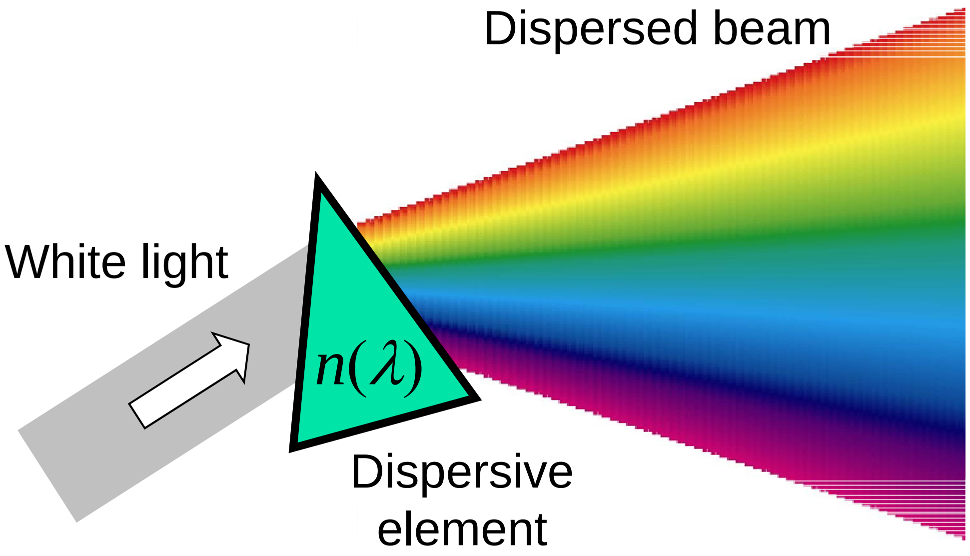
Practical numbers for material dispersion





Rainbow

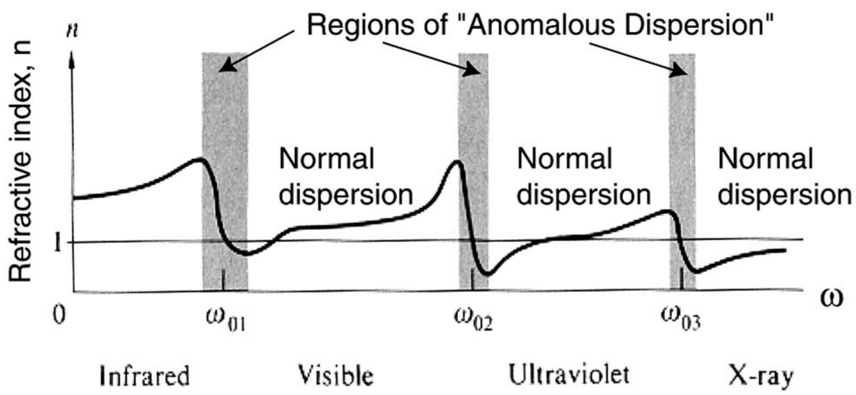
Dispersion of the refractive index allows prisms to separate white light into its components and to measure the wavelength of light.



Dispersion can be good or bad, depending on what you'd like to do.

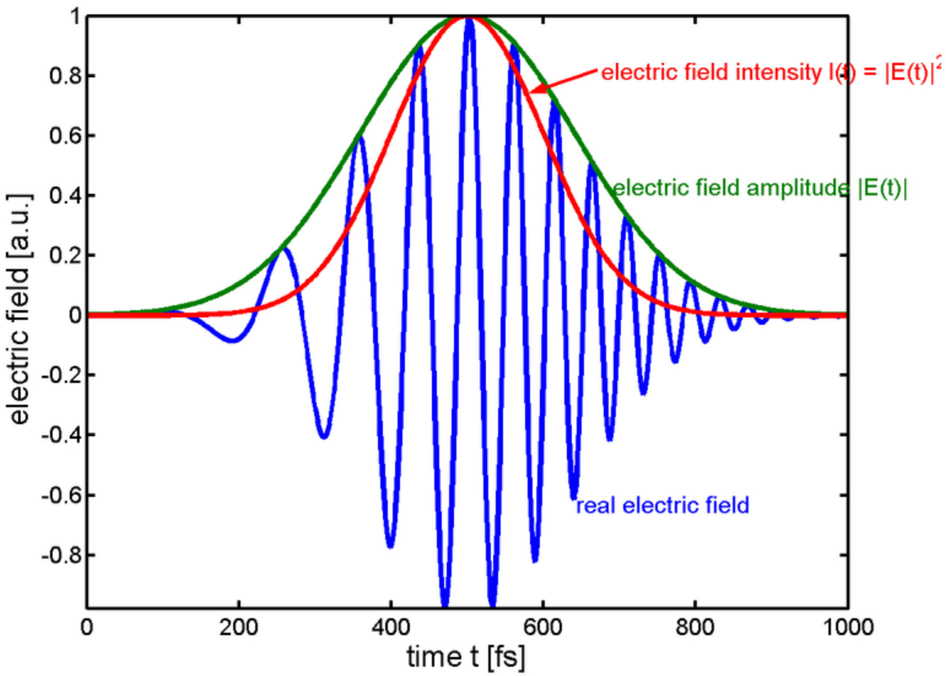
Dispersion: pulse chirping

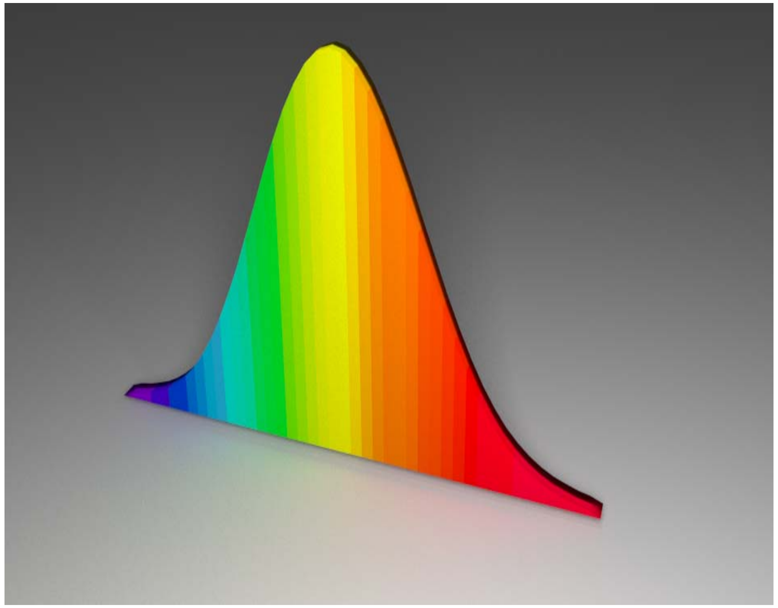
啁啾



Normal dispersion
n larger for higher frequency
$v_p = c/n$ ➔ "blue" travels slower

Anomalous dispersion
n smaller for higher frequency
$v_p = c/n$ ➔ "red" travels slower





Optical experimental data